Special Relativity Summary

- Lorentz Transformation: switching between different inertial frames
- Time Dilation and Length Contraction
- Velocity addition
- Mass, Energy, Momentum
- What quantities are invariant in all frames?
  - Space-time metric
  - Rest energy
- Transformation of Magnetic Fields

- Other topics:
  - Relativistic Doppler effect

Einstein's Postulates

1. The laws of mechanics and electromagnetism are the same for *all inertial reference frames*
   - Imagine a reference frame as a windowless room: one should not be able to tell whether the room is moving or stationary without looking outside
2. The velocity of light is independent of the motion of its source
   - Impossible to send out a light signal and then "catch up to it"
   - **Light has no rest frame**

Spacetime

$$0 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

- Imagine a flash of light at the origin at t=0
- Triggers spherical light wave going off in all directions
- Where will an observer be able to see the flash at time t?
- What if left hand side is not 0?
    - Gives us **lengths of spacetime intervals**
    - These lengths are invariant in all reference frames

34. In an inertial reference frame S, two events occur on the x-axis separated in time by Δt and in space by Δx. In another inertial reference frame S', moving in the x-direction relative to S, the two events could occur at the same time under which, if any, of the following conditions?

(A) For any values of Δx and Δt

(B) Only if $|\Delta x / \Delta t| < c$

(C) Only if $|\Delta x / \Delta t| > c$

(D) Only if $|\Delta x / \Delta t| = c$

(E) Under no condition

Lorentz Transformation

- Used to "boost" between different inertial frames
- Here, the boost means we switch into a frame moving at a velocity v along the x-direction

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - xv/c^2)$$

- Coordinates (t,x,y,z) ->(t',x',y',z')
- Primed frame is the moving frame, unprimed is "stationary"
- NB: only time coordinate and coordinates parallel to v
- NB: setting c=1, t-x transformations have exchange symmetry

Lorentz Transformation

- Recall: t-x symmetry in LT
- Which of these preserves the spacetime metric?

94. Which of the following is a Lorentz transformation? (Assume a system of units such that the velocity of light is 1.)

(A) $x' = 4x$
$y' = y$
$z' = z$
$t' = .25t$

(B) $x' = x - .75t$
$y' = y$
$z' = z$
$t' = t$

(C) $x' = 1.25x - .75t$
$y' = y$
$z' = z$
$t' = 1.25t - .75x$

(D) $x' = 1.25x - .75t$
$y' = y$
$z' = z$
$t' = .75t - 1.25x$

(E) None of the above

Time Dilation

- Processes appear to take *longer* when observed by a moving observer

$$t' = \gamma t$$

- t = time of process in stationary frame
- t' = time of process in moving frame (observer sits in a frame that moves at v relative to the stationary frame)
- **t' > t**
- Physical example:
  - Cosmic rays create muons high in the atmosphere. Muons decay very quickly in their own rest frame, relative to us the muons move so quickly that it seems to take much longer to decay. Hence we can observe and measure muons at sea level

Length Contraction

- Physical distances appear *shorter* when observed by a moving observer

$$L' = L/\gamma$$

- L’ = length measured by observer
- L = length in frame (moving relative to observer)
- **L’ < L**
- Physical example
  - Return to muon example: can also think of muon decay from the muon’s frame: the muon “sees” the world contract around it such that the distance from the atmosphere to the ground is smaller
  - No direct experimental evidence yet (difficult to make macroscopic objects move close to c)

Length Contraction

<u>Questions 69-71</u>

A car of rest length 5 meters passes through a garage of rest length 4 meters. Due to the relativistic Lorentz contraction, the car is only 3 meters long in the garage's rest frame. There are doors on both ends of the garage, which open automatically when the front of the car reaches them and close automatically when the rear passes them. The opening or closing of each door requires a negligible amount of time.

69. The velocity of the car in the garage's rest frame is

(A) 0.4 c
(B) 0.6 c
(C) 0.8 c
(D) greater than c
(E) not determinable from the data given

70. The length of the garage in the car's rest frame is

(A) 2.4 m
(B) 4.0 m
(C) 5.0 m
(D) 8.3 m
(E) not determinable from the data given

71. Which of the following statements is the best response to the question:

"Was the car ever inside a closed garage?"

(A) No, because the car is longer than the garage in all reference frames.
(B) No, because the Lorentz contraction is not a "real" effect.
(C) Yes, because the car is shorter than the garage in all reference frames.
(D) Yes, because the answer to the question in the garage's rest frame must apply in all reference frames.
(E) There is no unique answer to the question, as the order of door openings and closings depends on the reference frame.

Velocity Addition

- Take derivatives of Lorentz Transformation, being careful to distinguish between taking derivatives with respect to primed and unprimed time coordinates:

$$\dot{x}' = (\dot{x} - v) / \left(1 - \frac{\dot{x}\, v}{c^2}\right)$$

$$\dot{y}' = (\dot{y} / \gamma) / \left(1 - \frac{\dot{y}\, v}{c^2}\right)$$

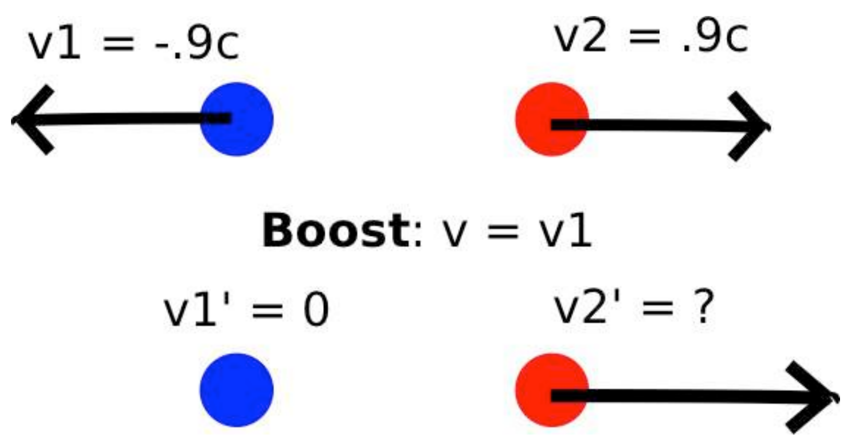
- dx'/dt' is velocity transformation parallel to direction of boost
- dy'/dt' is velocity transformation perpendicular to boost
- Check signs: if dx/dt = 0, dx'/dt' = -v in moving frame
- Note that it is impossible to get a velocity > c by changing frames

Velocity Addition

- We use this formula to find the sum of or difference between two velocities near c by boosting into the rest frame of one of the moving objects

$$\dot{x}' = (\dot{x} - v) / \left(1 - \frac{\dot{x}\, v}{c^2}\right)$$

- Example: How fast is red moving in blue's reference frame?



- Check signs: what if instead we boost into red's frame?

Velocity Addition Example

80. A tube of water is traveling at 1/2 *c* relative to the lab frame when a beam of light traveling in the same direction as the tube enters it. What is the speed of light in the water relative to the lab frame? (The index of refraction of water is 4/3.)

    (A) 1/2 *c*
    (B) 2/3 *c*
    (C) 5/6 *c*
    (D) 10/11 *c*
    (E) *c*

Velocity Addition Example

- In the rest frame of the tube, the light is only moving c/n = 3/4c<c, so let's just treat this like an ordinary velocity addition problem (c = 1):

$$v' = \frac{u+v}{1+uv}$$

- u is the velocity of the tube with respect to the lab frame
- v is the velocity of light within the tube

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(A) 1/2 *c*
(B) 2/3 *c*
(C) 5/6 *c*
(D) 10/11 *c*
(E) *c*

- Since u and v have the same sign in the lab frame, they have the same sign when using this formula (they would have opposite sign if the tube were moving in opposite direction of light

Energy and Rest Mass

$$p^2c^2 = E_{total}^2 - (m_{rest}c^2)^2$$

$$m_{rest}c^2 = \sqrt{E_{total}^2 - p^2c^2}$$

- Rest mass is constant in all frames (check this!)
- What happens to E when we pick the rest frame as our reference frame?

36. A lump of clay whose rest mass is 4 kilograms is traveling at three-fifths the speed of light when it collides head-on with an identical lump going the opposite direction at the same speed. If the two lumps stick together and no energy is radiated away, what is the mass of the composite lump?

(A) 4 kg
(B) 6.4 kg
(C) 8 kg
(D) 10 kg
(E) 13.3 kg

Energy and Momentum

- Energy (E/c) and momentum (p) transform like space-time coordinates:

$$p_x' = \gamma(p_x - vE/c^2)$$
$$p_y' = p_y$$
$$p_z' = p_z$$
$$E'/c = \gamma(E/c - pv/c)$$

- (One trick is to just set c = 1, v=β is a fraction of c)

Photons

- Photons
  - Photons have 0 rest mass
    - Travel at c
    - **No rest frame**
  - Photons do carry momentum
  - All energy comes from kinetic energy: **E = pc**

68. If a newly discovered particle X moves with a speed equal to the speed of light in vacuum, then which of the following must be true?

(A) The rest mass of X is zero.
(B) The spin of X equals the spin of a photon.
(C) The charge of X is carried on its surface.
(D) X does not spin.
(E) X cannot be detected.

Transformation of Electromagnetic Fields

- Thought experiment
  - Suppose I observe the electromagnetic fields of a stationary electron (electric field only)
  - Now I boost into a moving frame: electron appears to be moving relative to me (electric field *and* a magnetic field)
  - Where did the magnetic field come from?
  - Need the magnetic field for the first postulate to hold!
    - Check the net results of the two fields using test charges: do the forces on a test charge change?

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - \beta B_z) \qquad B'_y = \gamma(B_y + \beta E_z)$$

$$E'_z = \gamma(E_z + \beta B_y) \qquad B'_z = \gamma(B_z - \beta E_y)$$

Transformation of EM fields

49. The infinite xy-plane is a nonconducting surface, with surface charge density σ, as measured by an observer at rest on the surface. A second observer moves with velocity $v\hat{\mathbf{x}}$ relative to the surface, at height h above it. Which of the following expressions gives the electric field measured by this second observer?

(A) $\dfrac{\sigma}{2\epsilon_0}\,\hat{\mathbf{z}}$

(B) $\dfrac{\sigma}{2\epsilon_0}\sqrt{1 - v^2/c^2}\,\hat{\mathbf{z}}$

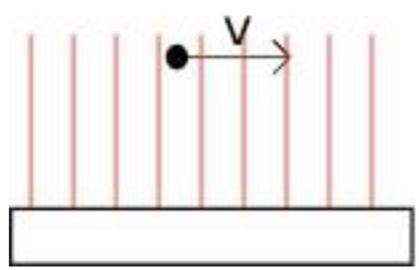
(C) $\dfrac{\sigma}{2\epsilon_0\sqrt{1 - v^2/c^2}}\,\hat{\mathbf{z}}$

(D) $\dfrac{\sigma}{2\epsilon_0}\left(\sqrt{1 - v^2/c^2}\,\hat{\mathbf{z}} + v/c\,\hat{\mathbf{x}}\right)$

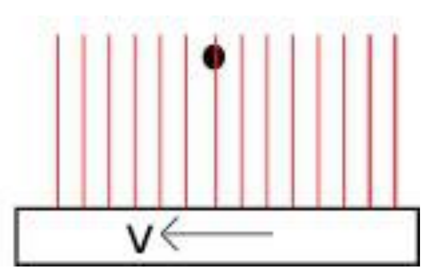
(E) $\dfrac{\sigma}{2\epsilon_0}\left(\sqrt{1 - v^2/c^2}\,\hat{\mathbf{z}} - v/c\,\hat{\mathbf{y}}\right)$

Transformation of EM fields

- Imagine drawing the electric field lines in the lab frame as being spaced evenly apart
- Now imagine boosting to the moving observer's frame
- Now the charged plane appears to be moving at -v**x**
- How do the distances between the evenly spaced electric field lines change in the moving frame?
  - Lengths contract, so field lines are *denser* and the electric field must appear *stronger* to the moving observer
  - Effectively, the observer observes the presence of more charge in a given amount of time if the plane is moving than if the plane is stationary



Lab frame



Observer's Frame

Transformation of EM fields

- The answer must reflect a stronger field
- Also, no symmetry breaking: the field is not bent in the y or x directions

- Can also remember how electromagnetic fields Lorentz transform under a boost in the x direction:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - \beta B_z) \qquad B'_y = \gamma(B_y + \beta E_z)$$

$$E'_z = \gamma(E_z + \beta B_y) \qquad B'_z = \gamma(B_z - \beta E_y)$$

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(B) $\frac{\sigma}{2\epsilon_0}\sqrt{1 - v^2/c^2}\ \hat{\mathbf{z}}$

(C) $\frac{\sigma}{2\epsilon_0\sqrt{1 - v^2/c^2}}\ \hat{\mathbf{z}}$

(D) $\frac{\sigma}{2\epsilon_0}\left(\sqrt{1 - v^2/c^2}\ \hat{\mathbf{z}} + v/c\ \hat{\mathbf{x}}\right)$

(E) $\frac{\sigma}{2\epsilon_0}\left(\sqrt{1 - v^2/c^2}\ \hat{\mathbf{z}} - v/c\ \hat{\mathbf{y}}\right)$

Other Topics

- Relativistic Doppler Effect

$$f' = f\sqrt{\frac{1+\beta}{1-\beta}}$$